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Linear stability of modulated circular Couette flow

By P. J. RILEY AND R. L. LAURENCE
Department of Chemical Engineering, University of Massachusetts, Amherst

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The linear stability of modulated circular Couette flow to axisymmetric disturbances is examined in the narrow-gap limit. The outer cylinder is assumed stationary, while the inner is modulated both with and without a mean rotation. The equations governing the disturbance motion are solved by a Galerkin expansion with time-dependent coefficients, and the stability of the motion determined by Floquet theory. Modulation is found, in general, to destabilize the flow due to steady rotation, although weak stabilization is found for some modulation amplitudes at intermediate frequencies.

1. Introduction

There has been a growing interest in the literature in the stability of time-dependent flows. The results are fairly extensive and are reviewed by Davis (1976) and Riley (1975).

We are concerned with the stability of the time-periodic flow between concentric circular cylinders when the outer cylinder is at rest and the inner has angular velocity

$$\Omega(t) = \Omega_m + \Omega_p \cos \omega' t'.$$

The problem with Ω_m and Ω_p both non-zero has been investigated experimentally by Donnelly (1964). His results have provided considerable impetus to theoretical investigations of time-periodic motions and, indeed, have been used as a 'standard' for theoretical studies of the stability of time-dependent motions and configurations. Donnelly found that modulation stabilizes circular Couette flow at low frequencies $0{\cdot}3 \leqslant \gamma \leqslant 1{\cdot}0$ and for small modulation amplitudes

$$0 < \Omega_p/\Omega_m \leqslant 0{\cdot}25 \quad \text{where} \quad \gamma = (\omega' d^2/2\nu)^{\frac{1}{2}},$$

ν is the kinematic viscosity, ω' the angular frequency and d the gap width. Some evidence of destabilization was found at higher frequencies $\gamma > 1{\cdot}0$. He performed experiments at four gap widths. At one intermediate value, optimum stabilization occurred, for all modulation amplitudes investigated, at a frequency $\gamma = (2{\cdot}7)^{-1}$. Stabilization was found to be enhanced by increasing modulation amplitude, $\Omega_p/\Omega_m < 0{\cdot}25$. At the other gap widths stabilization increased with decreasing frequency. Recently Homsy (1974) has suggested that the data of Donnelly are a poor comparison for theory; this point will be discussed further in §4.1.

Conrad & Criminale (1965) present an energy analysis of the stability of circular Couette flow to axisymmetric disturbances in the narrow-gap limit. They treat the system studied by Donnelly as one of many cases. Several aspects of their work are questionable. They find weak stabilization for small amplitude ratios, in agreement with the data of Donnelly, but the energy bound for the modulated problem is certainly less than that for the steady flow at low frequencies; see §4.1. They find for modulation of the outer cylinder with non-zero mean rotation (the inner cylinder stationary) that a Reynolds number, based on the maximum velocity of the outer cylinder, tends to zero as frequency increases. This appears to be a contradiction of a theorem of Serrin (1960). They present results for the steady problem and make a comparison with the data of Taylor (1923) and the energy bounds of Serrin (1959). Their calculation is based, however, on radii of $R_1 = 3{\cdot}8$ cm and $R_2 = 4{\cdot}035$ cm for the inner and outer cylinder respectively, while the results of Taylor and Serrin are for $R_1 = 3{\cdot}55$ cm and $R_2 = 4{\cdot}035$ cm. Re-evaluation of their criterion shows that it is particularly sensitive to gap width and is in agreement neither with the results of Serrin nor with the data of Taylor.

Meister (1963) considers the linear stability of circular Couette flow with arbitrary time dependence to axisymmetric disturbances in the narrow-gap limit. He shows that a Galerkin expansion with time-dependent coefficients converges to a generalized solution of the governing equations. Meister & Münzer (1966), making the same assumptions as Meister (1963), study the stability of time-periodic circular Couette flow generated by steady rotation of both cylinders with a modulation superimposed on the inner one. They find that the disturbance kinetic energy increases less rapidly in the presence of modulation, and claim, therefore, that modulation is stabilizing. This may be inconclusive, however, as they appear to determine disturbance growth over less than one-tenth of a period and the maximum angular velocity of the inner cylinder is not attained. The long-term behaviour is thus unclear.

Thompson (1968) reports critical data for modulation of the inner cylinder, with zero mean, both with and without steady rotation of the outer. Outer rotation is found to stabilize inner modulation. He remarks that modulation superimposed on steady rotation of the inner cylinder (with the outer stationary) is destabilizing. Qualitatively he found that at low frequencies the flow becomes unstable as soon as the maximum angular velocity exceeds the critical angular velocity Ω_0 for the corresponding steady flow, i.e. $\Omega_m + \Omega_p \geqslant \Omega_0$. At higher frequencies instability occurs when the mean angular velocity exceeds the critical value for the steady flow, i.e. $\Omega_m \geqslant \Omega_0$. These findings are inconsistent with the instability data of Donnelly (1964), but in agreement with the existence of the 'transient vortices' which he reported.

Thompson also performed a linear stability analysis for modulation of the inner cylinder with zero mean. Disturbances were assumed axisymmetric and axially periodic. Radial spatial dependence was removed by a finite-difference formulation followed by numerical integration in time. Neutral stability was determined by observing the growth of disturbance kinetic energy over several cycles. The theory was found to be in good agreement with his experimental data for the range investigated: $1{\cdot}4 < \gamma < 4{\cdot}0$.

Recently Hall (1975, 1976) treated this problem using asymptotic methods.

The results are largely complementary to those presented in this work and will be discussed in detail in §4.

In the following we examine theoretically the experimental configurations of Donnelly (1964) and Thompson (1968); viz. the outer cylinder of a circular Couette apparatus is stationary and the inner modulated both with and without a mean rotation. In §2 we formulate the linearized stability problem in the narrow-gap limit and describe the solution method. In §3 we present the theoretical results for the particular cases considered, and in §4 make a comparison with the experimental data available and earlier theoretical work. In §5 we briefly survey the results and make some concluding remarks.

2. Mathematical formulation

2.1. *Primary flow*

We consider the flow of a Newtonian incompressible fluid of density ρ and viscosity μ between infinitely long concentric cylinders having radii R_1 and R_2. As in the work of Hall (1975, 1976), it is assumed that the separation d between cylinders is small compared with the radius R_1 of the inner cylinder. We make the usual small-gap approximation, so convenient length, time and velocity scales are

$$d, \quad \rho d^2/\mu, \quad R_1\bar{\Omega} \tag{1}$$

and the primary flow $\mathbf{U}'(r,t) = (0, V'(r,t), 0)$ is adequately represented by $\mathbf{U}(x,\tau) = (0, V(x,\tau), 0)$, the solution of

$$\partial V/\partial \tau = \partial^2 V/\partial x^2, \tag{2a}$$

$$V(0,\tau) = s + \epsilon \cos \omega\tau, \tag{2b}$$

where

$$x = (r-R_1)/d, \quad \omega = \rho\omega' d^2/\mu, \quad V(x,\tau) = V'(r,t)/R_1\bar{\Omega},$$

$$s = \Omega_m/\bar{\Omega}, \quad \epsilon = \Omega_p/\Omega_m, \quad \tau = \mu t/\rho d^2.$$

The solution may be written as

$$V(x,r) = s(1-x) + \epsilon\{f_1(x)\cos\omega\tau + f_2(x)\sin\omega\tau\}, \tag{3}$$

where

$$f_1(x) = \{g_1(0)\,g_1(x) + g_2(0)\,g_2(x)\}/W(\gamma),$$

$$f_2(x) = \{g_2(0)\,g_1(x) + g_1(0)\,g_2(x)\}/W(\gamma),$$

$$g_1(x) = \sinh\tilde{\lambda}(x)\cos\tilde{\lambda}(x), \quad g_2(x) = \cosh\tilde{\lambda}(x)\sin\tilde{\lambda}(x),$$

$$\tilde{\lambda}(x) = \gamma(1-x), \quad W(\gamma) = g_1^2(0) + g_2^2(0).$$

$\gamma = (\frac{1}{2}\omega)^{\frac{1}{2}}$ is the reciprocal of the dimensionless Stokes-layer thickness, and will be referred to as 'frequency'.

At low frequencies,

$$V(x,\tau) \simeq (1-x)(s + \epsilon\cos\omega\tau), \tag{4}$$

while at high frequencies,

$$V(x,\tau) \simeq s(1-x) + \epsilon e^{-\gamma x}\cos(\omega\tau - \gamma x). \tag{5}$$

Hence at high frequencies the Stokes layer is confined to a region adjacent to the inner cylinder, which becomes increasingly narrow as the frequency increases. The complete profile (3) was used for all calculations.

2.2. *Stability equations*

We suppose that the primary velocity and pressure fields $\mathbf{U} = (0, V(r,\tau), 0)$ and $\Pi(r,\tau)$ are disturbed:

$$\left.\begin{aligned} \mathbf{U}^*(r,\theta,z;\tau) &= \mathbf{U}(r,\tau)+\tilde{\mathbf{u}}(r,\theta,z;\tau), \\ \Pi^*(r,\theta,z;\tau) &= \Pi(r,\tau)+\tilde{p}(r,\theta,z,\tau), \end{aligned}\right\} \quad (6)$$

where $\mathbf{U}$ is $(2\pi/\omega)$-periodic in time and (r,θ,z) are cylindrical co-ordinates.

The difference variables $\tilde{\mathbf{u}}$ and $\tilde{p}$ satisfy

$$\mathscr{R}^{-1}\,\partial\mathbf{u}/\partial\tau+\tilde{\mathbf{u}}.\nabla\tilde{\mathbf{u}}+\mathbf{U}.\nabla\tilde{\mathbf{u}}+\tilde{\mathbf{u}}.\nabla\mathbf{U} = -\nabla p+\mathscr{R}^{-1}\Delta\tilde{\mathbf{u}}, \quad (7a)$$

$$\nabla.\tilde{\mathbf{u}} = 0, \quad (7b)$$

$$\tilde{\mathbf{u}} = 0 \quad \text{on solid boundaries}, \quad (7c)$$

$$\tilde{\mathbf{u}}(x,\theta,z;0) = \tilde{\mathbf{u}}_0, \quad (7d)$$

where the reference scales are given by (1), with $z = z'/d$ and the Reynolds number

$$\mathscr{R} = R_1\bar{\Omega}d/\nu.$$

In the present study we linearize (7) and solve

$$\mathscr{R}^{-1}\partial\tilde{\mathbf{u}}/\partial\tau+\mathbf{U}.\nabla\tilde{\mathbf{u}}+\tilde{\mathbf{u}}.\nabla\mathbf{U} = -\nabla\tilde{p}+\mathscr{R}^{-1}\Delta\tilde{\mathbf{u}}, \quad (8a)$$

$$\nabla.\tilde{\mathbf{u}} = 0, \quad (8b)$$

$$\tilde{\mathbf{u}} = 0 \quad \text{on solid boundaries}, \quad (8c)$$

$$\tilde{\mathbf{u}}(r,\theta,z;0) = \tilde{\mathbf{u}}_0. \quad (8d)$$

Linearization for periodic motions is justified by the work of Yudovitch (1970). The critical Reynolds number determined, $\mathscr{R}_c$, is a sufficient condition for instability, i.e. if the motion satisfies $\mathscr{R} < \mathscr{R}_c$, then there is at least one disturbance which will grow and lead to a different asymptotic motion. Yudovitch shows that it is not necessary to solve (8) as an initial-value problem. We may instead look for solutions of the form

$$\tilde{\mathbf{u}}(\mathbf{r},\tau) = e^{\sigma\tau}\mathbf{u}_*(r,\tau), \quad (9)$$

where $\mathbf{u}_*$ is $(2\pi/\omega)$-periodic in τ, and solve the following eigenvalue problem:

$$\mathscr{R}^{-1}(\sigma\mathbf{u}_*+\partial\mathbf{u}_*/\partial\tau)+\mathbf{u}_*.\nabla\hat{\mathbf{U}}+\hat{\mathbf{U}}.\nabla\mathbf{u}_* = -\nabla p_*+\mathscr{R}^{-1}\Delta\mathbf{u}_*, \quad (10a)$$

$$\nabla.\mathbf{u}_* = 0, \quad (10b)$$

$$\mathbf{u}_*(\mathbf{r},\tau+2\pi/\omega) = \mathbf{u}_*(\mathbf{r},\tau), \quad (10c)$$

$$\mathbf{u}_* = 0 \quad \text{on solid boundaries}. \quad (10d)$$

The stability of the motion is determined by the eigenvalues $\{\sigma_i\}$ of (10). If the corresponding eigenfunctions $\{\mathbf{u}_{*,i}(\mathbf{r},\tau)\}$ are complete in an appropriate sense, the solution of problem (8), corresponding to an arbitrary infinitesimal initial disturbance $\tilde{\mathbf{u}}_0$, may be constructed using an eigenfunction expansion. It follows that the stability criterion is independent of the exact form of the initial disturbance in the same sense as for a linear stability analysis of a steady flow.

In the following we do not solve (10) directly, but treat (8) by Galerkin's method and determine the $\{\sigma_i\}$ by the application of Floquet theory.

For modulated circular Couette flow in the narrow-gap limit, Galerkin's method is known to converge if the initial data $\tilde{\mathbf{u}}_0$ are sufficiently smooth (Meister 1963). The solution is then of necessity a linear combination of terms of the form (9). It follows that the stability criterion determined by this method is the one required by the Lyapounov theorem of Yudovitch (1970). Furthermore, we may assume that the set $\{\mathbf{u}_{*,i}(r,\tau)\}$ is complete.

We suppose the disturbances $\tilde{\mathbf{u}}$ and $\tilde{p}$ to be axially periodic and axisymmetric:

$$\left.\begin{aligned} \tilde{u}_r &= u(r,\tau)\cos\alpha z, & \tilde{u}_z &= w(r,\tau)\sin\alpha z, \\ \tilde{u}_\theta &= v(r,\tau)\cos\alpha z, & \tilde{p} &= p(r,\tau)\cos\alpha z, \end{aligned}\right\} \tag{11}$$

where α is the dimensionless axial wavenumber. The formulation used is the usual procedure outlined in Chandrasekhar (1961, p. 294).

The most dangerous disturbance and the corresponding asymptotic secondary flow are axisymmetric for steady flow, i.e. $\epsilon = 0$ (see Kruger, Gross & DiPrima 1966). It is not clear that disturbances with non-zero azimuthal wavenumber will not be important for the modulated problem. It was decided, however, to restrict attention to axisymmetric disturbances, not only to avoid the 'curse of dimensionality', but also because Thompson (1968) does not report having observed any non-axisymmetric disturbances.

The eventual form of the disturbance equations, using the small-gap approximation, and appropriate scaling† of the velocity components, yields the equations

$$\mathscr{L}\partial u/\partial\tau = \mathscr{L}^2 u - \alpha\mathscr{R}(2\delta)^{\frac{1}{2}}\, Vv, \tag{12a}$$

$$\partial v/\partial\tau = Lv - \alpha\mathscr{R}(2\delta)^{\frac{1}{2}}\,(DV)\,u, \tag{12b}$$

$$\left.\begin{aligned} u = Du = v = 0 \quad &\text{at} \quad x = 0, 1, \\ u(x,0) = u_0, \quad &v(x,0) = v_0, \end{aligned}\right\} \tag{12c}$$

where $\mathscr{L} = D^2 - \alpha^2$, and $\delta = d/R_1$.

For modulation with zero mean ($\Omega_m = 0$), let $\overline{\Omega} = \Omega_p$; then $s = 0$ and $\epsilon = 1$ in (3), and we define $\mathscr{R} = \mathscr{R}_1\Omega_p\, d/\nu = \mathscr{R}_p$. For modulation with non-zero mean we set $\overline{\Omega} = \Omega_m$, hence $\mathscr{R} = R_1\Omega_m d/\nu = \mathscr{R}_m$ and, in (3), $\epsilon = \Omega_p/\Omega_m$ is the dimensionless modulation amplitude.

2.3. *Solution method*

We expand the solution in a complete set of functions which satisfy the boundary conditions (12*c*):

$$u = \sum_{j=1}^{N} a_j(\tau; N)\,\phi_j(x), \quad v = \sum_{j=1}^{\tilde{N}} b_j(\tau; \tilde{N})\,\tilde{\phi}_j(x), \tag{13a, b}$$

† The scaling used here results in the characteristic number $\mathscr{R}(2\delta)^{\frac{1}{2}}$. This is related to the usual Taylor number by $T = 2\mathscr{R}^2\delta$.

where $$\phi_j(x) = x^2(1-x)^2 \mathscr{J}_{j-1}(x), \quad \tilde{\phi}_j(x) = x(1-x)\,\tilde{\mathscr{J}}_{j-1}(x). \tag{14}$$

$\mathscr{J}_j$ and $\tilde{\mathscr{J}}_j$ are Jacobi polynomials of order $j-1$ and are defined in the appendix. N and $\tilde{N}$ are integers to be determined by requirements of accuracy; we take $N = \tilde{N}$. In what follows, we shall suppress the explicit dependence of the coefficients $a_j(\tau; N)$ and $b_j(\tau; N)$ and the resulting eigenvalues upon N.

The errors ϵ_N and $\tilde{\epsilon}_N$ resulting in (12) from truncation of the infinite expansions to N terms are forced to be orthogonal to each element of the respective set of approximating functions $\{\phi_j\}_{j=1}^N$ and $\{\tilde{\phi}_i\}_{j=1}^N$, giving rise to a system of $2N$ ordinary differential equations for the coefficients $\{a_j(\tau)\}_{j=1}^N$ and $\{b_j(\tau)\}_{j=1}^N$. This may be written as

$$\left.\begin{aligned}\begin{bmatrix}\dot{\mathbf{a}}\\ \dot{\mathbf{b}}\end{bmatrix} &= \mathbf{G}(\tau; \gamma, \epsilon, \alpha, \mathscr{R}\sqrt{\delta})\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix},\\ \mathbf{a}(0) = \mathbf{a}_0, &\quad \mathbf{b}(0) = \mathbf{b}_0,\end{aligned}\right\} \tag{15}$$

where a dot denotes differentiation with respect to τ, and $\mathbf{G}(\tau)$ is a $(2\pi/\omega)$-periodic, $2N \times 2N$ square matrix defined in the appendix.

We construct a fundamental matrix $\mathbf{X}(\tau)$ of the system (15):

$$\dot{\mathbf{X}} = \mathbf{G}\mathbf{X}, \quad \mathbf{X}(0) = \mathbf{I} \tag{16}$$

(where $\mathbf{I}$ is the $2N \times 2N$ identity matrix). Floquet theory (Coddington & Levinson 1955, p. 78) states that $\mathbf{X}(\tau)$ may be represented as

$$\mathbf{X}(\tau) = \mathbf{P}(\tau)\exp(\tau\mathbf{C}),$$

where $\mathbf{P}(\tau)$ is $(2\pi/\omega)$-periodic matrix, $\mathbf{C}$ is a constant matrix and both may be complex.

The stability of the motion is determined by the eigenvalues $\{\sigma_j\}_{j=1}^N$ of the matrix $\mathbf{C}$, which are easily determined from the eigenvalues $\{\lambda_j\}_{j=1}^N$ of $\mathbf{X}(2\pi/\omega)$:

$$\mathbf{X}\left(\frac{2\pi}{\omega}\right) = \mathbf{P}\left(\frac{2\pi}{\omega}\right)\exp\left(\frac{2\pi}{\omega}\mathbf{C}\right) = \mathbf{P}(0)\exp\left(\frac{2\pi}{\omega}\mathbf{C}\right) = \mathbf{I}\exp\left(\frac{2\pi}{\omega}\mathbf{C}\right).$$

If $\mathbf{C}$ is transformed to Jordan canonical form by a matrix $\mathbf{Q}$, then $\mathbf{X}(2\pi/\omega)$ is transformed to upper triangular form by $\mathbf{Q}$. Hence

$$\lambda_j = \exp\left(\frac{2\pi}{\omega}\sigma_j\right) = \exp\left(\frac{2\pi}{\omega}\sigma_j + 2\pi k i\right), \quad k = 0, 1, 2, \ldots,$$

and $$\sigma_j = \omega\{\ln\lambda_j \pm 2\pi k i\}/2\pi.$$

To calculate the eigenfunctions $u_{*,j}$ and $v_{*,j}$ associated with the eigenvalue σ_j we have only to determine the corresponding initial conditions $\mathbf{a}_0^j$, $\mathbf{b}_0^j$ for system (15), which satisfy

$$\mathbf{X}(2\pi/\omega)\begin{bmatrix}\mathbf{a}_0^j\\ \mathbf{b}_0^j\end{bmatrix} = \lambda_j\begin{bmatrix}\mathbf{a}_0^j\\ \mathbf{b}_0^j\end{bmatrix}.$$

The coefficients for the Galerkin expansion are

$$\begin{bmatrix}\mathbf{a}^j(\tau)\\ \mathbf{b}^j(\tau)\end{bmatrix} = \mathbf{X}(\tau)\begin{bmatrix}\mathbf{a}_0^j\\ \mathbf{b}_0^j\end{bmatrix} = \mathbf{P}(\tau)\exp(\tau\mathbf{C})\begin{bmatrix}\mathbf{a}_0^j\\ \mathbf{b}_0^j\end{bmatrix} = \mathbf{P}(\tau)\exp(\tau\sigma_j)\begin{bmatrix}\mathbf{a}_0^j\\ \mathbf{b}_0^j\end{bmatrix}, \tag{17}$$

and $$u_{*,j} = \exp(-\tau\sigma_j)\,u(x,\tau), \quad v_{*,j} = \exp(-\tau\sigma_j)\,v(x,\tau),$$

where $u(x,\tau)$ and $v(x,\tau)$ are given by (13) with the coefficients determined by (17).

Let σ_1 be the eigenvalue with greatest real part. The primary motion is stable or unstable to infinitesimal disturbances according as $\mathrm{Re}\,(\sigma_1) \lesssim 0$. The boundary between conditional stability and certain instability is given by $\mathrm{Re}\,(\sigma_1) = 0$.

The finite amplitude motion which develops in the neighbourhood of bifurcation has been considered by Joseph (1972) and Iooss (1972). At neutral stability, $Re\,(\sigma_1) = 0$, let $\omega_1 = \mathrm{Im}\,(\sigma_1)$. If $\omega_1 = 0$, a finite amplitude motion with period $2\pi/\omega$ will arise (Joseph 1972); furthermore, supercritical branches are stable and subcritical unstable. If $\omega_1/\omega = m/n$ is rationally dependent and m/n is irreducible, motion of period $2\pi/\omega$ will develop (Iooss 1972). It is to be expected that supercritical branches will be stable and subcritical unstable (cf. Ruelle & Takens 1971). If ω_1/ω is rationally independent the quasi-periodic motion may not survive nonlinear interactions, which might lead to motion of quite different character (Joseph 1972).

2.4. *Numerical technique*

The matrix inversions required to derive system (15) and the calculation of the eigenvalues of $\mathbf{X}(2\pi/\omega)$ were performed using programs provided by the University of Massachusetts Computing Center. The inversion routine uses Gauss–Jordan elimination with complete pivoting. The eigenvalue routine uses the QR algorithm (Francis 1961; Wilkinson 1965). It was found in all cases that

$$\mathrm{tr}\,\mathbf{X}\,(2\pi/\omega) = \sum_{j=1}^{2N} \lambda_j \quad \text{to at least 9 decimal places.}$$

The fundamental matrix $\mathbf{X}(\tau)$ of system (16) was constructed with a fifth-order Runge–Kutta–Butcher formula (Lapidus & Seinfeld, 1971, p. 61), using double precision. A check on the numerical integration is provided by

$$\det \mathbf{X}\left(\frac{2\pi}{\omega}\right) = \prod_{j=1}^{2N} \lambda_j = \exp\left\{\int_0^{2\pi/\omega} \mathbf{G}(s)\,ds\right\},$$

where det and tr denote determinant and trace respectively (see Coddington & Levinson 1955, p. 82). Alternatively

$$\sum_{j=1}^{2N} \mathrm{Re}\,(\sigma_j) = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} \mathrm{tr}\,\mathbf{G}(s)\,ds = \mathrm{tr}\,\mathbf{G},$$

$$\sum_{=1}^{2N} \mathrm{Im}\,(\sigma_j) = \pm\, 2\pi k i, \quad k = 0, 1, 2, \ldots.$$

System (15) is reasonably stiff. It is difficult to calculate accurately the values of those eigenvalues having large negative real parts if $\omega = 2\gamma^2$ is small or N large. For example, let

$$\xi = \mathrm{tr}\,\mathbf{G}\Big/\sum_{j=1}^{2N} \mathrm{Re}\,(\sigma_j).$$

At $\gamma = 3{\cdot}0$, it was found that $\xi = 1{\cdot}5$ for $N = 3$ and $\xi = 2{\cdot}7$ for $N = 5$. At higher frequencies, $\gamma \geqslant 5{\cdot}0$ for $N = 3$ and $\gamma \geqslant 8{\cdot}0$ for $N = 5$, $\xi = 1{\cdot}0$ with an accuracy of better than $0{\cdot}5\,\%$.

The steady problem was solved using Floquet theory by forming a fundamental matrix $\mathbf{X}(\tau)$ at different instants of time τ_*. It was found that $\xi = \xi(\gamma, N)$, when $\tau_* = 2\pi/\omega = \pi/\gamma^2$. However, the critical parameters were independent of τ_* as required. Thus the largest eigenvalue found is accurate, but the values of higher eigenvalues are not reliable if the period of oscillation is large. For fixed N, $\xi(\gamma, N)$ is a monotonically decreasing smooth function of γ asymptotic to unity, and any departure from this behaviour proves to be a very sensitive check on the stability of the integration.

Critical parameters are determined by finding for fixed ϵ, γ and N values α_c and $\mathscr{R}_c\sqrt{\delta}$ such that

$$|\mathrm{Re}\,\{\sigma_1(\epsilon, \gamma; \alpha_c, \mathbf{R}_c\sqrt{\delta})\}| < 10^{-4}, \tag{18a}$$

$$\mathscr{R}_c\sqrt{\delta} = \mathscr{R}(\alpha_c)\sqrt{\delta} = \min \mathscr{R}(\alpha)\sqrt{\delta}. \tag{18b}$$

The two-dimensional search in the α, $\mathscr{R}\sqrt{\delta}$ plane was carried out using two one-dimensional searches. For fixed α, a value $\mathscr{R}\sqrt{\delta}$ such that (18a) was satisfied was found using the secant method (Ralston 1965, chap. 2). For fixed $\mathscr{R}\sqrt{\delta}$, Re (σ_1) was maximized as a function of α using an accelerated search with quadratic fitting as a predictor (Jacoby, Kowalik & Pizzo 1972, p. 69). The search was terminated when the increment size $\Delta\alpha$ before the next prediction satisfied $|\Delta\alpha| < 3{\cdot}0 \times 10^{-3}$. The procedure was then repeated until (18) were satisfied with the accuracy required. In particular, condition (18a) is sufficiently stringent that values of $\mathscr{R}_c\sqrt{\delta}$ are found accurately to three decimal places for the values of N chosen.

At low frequencies, $\gamma \leqslant 2{\cdot}0$, an excessive amount of computer time is required to undertake an extensive search using the fundamental-matrix approach. However, since the magnitudes of the real part of the eigenvalues vary considerably, the projection corresponding to the eigenvalue σ_1 is quickly dominant.

Initial values for $\mathbf{a}_0$ and $\mathbf{b}_0$ were produced by a random-number generator (RANF, RANFSET from the CDC 3600 Fortran Library Programs), and the disturbance kinetic energy

$$K(\tau) = \int_0^1 \tfrac{1}{2}(u^2+v^2+w^2)\,dx$$

calculated. The calculations above were performed using the condition

$$\left|\ln\left\{\frac{K(4\pi/\omega)}{K(2\pi/\omega)}\right\}\right| < 10^{-4}, \tag{19}$$

rather than (18a). This technique provided very accurate initial estimates for the application of the fundamental-matrix algorithm; indeed, the results were identical. At higher frequencies the fundamental-matrix method was used directly.

3. Results and discussion

3.1. *Precision of the numerical procedure*

An analysis of the steady problem with the outer cylinder stationary and the inner one rotating was used as an initial test of the calculation procedure. The

		$\gamma = 1{\cdot}0,\ \epsilon = 0{\cdot}25$		
N	α	$\mathscr{R}_m\surd\delta$	$[\Delta\alpha]$	$[\Delta(\mathscr{R}_m\surd\delta)]$
3	3·118	41·120	—	—
4	3·119	41·100	$-3{\cdot}2\times10^{-4}$	$4{\cdot}9\times10^{-4}$
5	3·119	41·091	0	$2{\cdot}2\times10^{-4}$
6	3·119	41·091	0	0
		$\gamma = 3{\cdot}75,\ \epsilon = 2{\cdot}0$		
3	3·053	43·173	—	—
4	3·077	42·922	$-7{\cdot}8\times10^{-3}$	$5{\cdot}8\times10^{-3}$
5	3·083	42·849	$-1{\cdot}9\times10^{-3}$	$1{\cdot}7\times10^{-3}$
6	3·086	42·827	$-9{\cdot}7\times10^{-4}$	$5{\cdot}1\times10^{-4}$
		$\gamma = 13{\cdot}0,\ \epsilon = \infty$		
N	α	$\mathscr{R}_p\surd\delta$	$[\Delta\alpha]$	$[\Delta(\mathscr{R}_p\surd\delta)]$
7	11·157	740·835	—	—
8	11·072	730·7	$7{\cdot}7\times10^{-3}$	$1{\cdot}39\times10^{-2}$
9	11·063	721·499	$8{\cdot}1\times10^{-4}$	$1{\cdot}28\times10^{-2}$

TABLE 1. Variation of accuracy of representation as a function of ϵ and γ. $[\Delta Z_i] = (Z_{i-1}-Z_i)/Z_i$.

results may be compared with those reported by Chandrasekhar (1961, p. 304) and Hall (1975). Chandrasekhar gives $\mathscr{R}_0\surd\delta = 41{\cdot}172$ and $\alpha_0 = 3{\cdot}12$ for the order of approximation $N = 3$. We found $\mathscr{R}_0\surd\delta = 41{\cdot}198$ and $\alpha_0 = 3{\cdot}1265$ for $N = 3$ and $\mathscr{R}_0\surd\delta = 41{\cdot}170$ and $\alpha_0 = 3{\cdot}1265$ for $N = 5$. Hall gives $\mathscr{R}_0\surd\delta = 41{\cdot}170$ and $\alpha_0 = 3{\cdot}1266$.

A more detailed analysis of the convergence of the numerical procedure was then performed for the time-dependent flow problem. The calculations clearly indicated that the degree of the approximation N required to maintain precision varied strongly with the parameters ϵ and γ. Computation time increased significantly with increasing N. The reported results are therefore not uniformly precise. Table 1 shows how the critical values of $\mathscr{R}_0\surd\delta$ and α change with N, ϵ and γ. The estimated maximum error in both the critical Taylor number and the critical wavenumber is less than 0·6 %, except for $\epsilon = 10{\cdot}0$ and $\gamma > 6$ and for modulation with zero mean with $\gamma > 8$, where the estimated maximum error is less than 1·5%.

In the course of the calculations, the order of the approximation was changed to minimize computational time while maintaining adequate precision. For low frequencies, $\gamma < 2{\cdot}0$, the velocity profile is relatively simple and $N = 3$ was found sufficient. For modulation with non-zero mean the oscillatory shear flow has a boundary-layer character at high frequencies and does not substantially affect stability, and $N = 3$ was again adequate. For intermediate frequencies, $N = 5$ or 6 was required. For modulation with zero mean the value of N required for a good approximation increases with frequency; for example, at $\gamma = 11{\cdot}0$, $N = 9$ gives results estimated to be precise to within 1 %. The convergence of the wavenumber with increasing N was sometimes oscillatory. If the response was synchronous ($\omega_1 = 0$) $\mathscr{R}_c\surd\delta$ was a monotonically decreasing function of N, but if the response was half-frequency ($\omega_1 = \frac{1}{2}\omega$) convergence was sometimes oscillatory.

The results of the calculations are presented graphically, thus the variation of accuracy is not apparent. Tabulated values are available from the authors upon request.

For the modulated flow the computing time, at fixed N, is proportional to γ^{-2}. For $\gamma = 3{\cdot}0$ and $N = 3{\cdot}0$, approximately 30 min of CDC 3600 computer time are required to evaluate the critical parameters for $\epsilon = 0{\cdot}1$, $0{\cdot}5$, $1{\cdot}0$, $2{\cdot}0$, $5{\cdot}0$ and $10{\cdot}0$. As γ becomes small the computing time increases; indeed, as stated above, it becomes impracticable to perform a complete search using the fundamental-matrix technique. However, calculations using condition (19) are relatively fast, and for $\gamma < 1{\cdot}0$ the critical parameters are only weak functions of frequency. For flows with non-zero mean the fundamental-matrix technique is fast for $\gamma > 3{\cdot}0$ (i.e. 10–20 min to evaluate the critical parameters for $\epsilon = 1{\cdot}0$–$5{\cdot}0$ at a single frequency) even though $N = 5$ or 6 may be required. For the modulated flows with zero mean the value of N required for accurate simulation increases monotonically with γ, and computing time becomes excessive. However, as will be discussed in §3.2 an asymptotic region is reached and calculations are not required at higher frequencies unless the eigenfunctions need be constructed. The major time-consuming step is the numerical integration of system (16). The Runge–Kutta scheme used here does not have a large stability region but it is accurate (Lapidus & Seinfeld 1971). A saving in computer time may be possible using schemes specifically developed for the integration of stiff systems.

3.2. *Modulation of Couette flow with zero mean*

The first case studied was modulated Couette flow with zero mean. The critical Taylor number $\mathscr{R}_p\sqrt{\delta}$ and wavenumber α are shown in figures 1 and 2 respectively for $1{\cdot}0 \leqslant \gamma \leqslant 13{\cdot}0$, and for $\gamma \leqslant 1{\cdot}0$ in figures 4 and 6 (figure 4 gives $\overline{\mathscr{R}}\sqrt{\delta}$ *vs.* γ, where $\overline{\mathscr{R}} \equiv R_1(\Omega_m^2+\Omega_p^2)^{\frac{1}{2}}\,d/\nu$; the curve for $\epsilon = \infty$ corresponds to modulation with zero mean, $\overline{\mathscr{R}}\sqrt{\delta} = \mathscr{R}_p\sqrt{\delta}$).

From figures 4 and 6, the results for low frequency, $\gamma \leqslant 1{\cdot}0$, are most independent of frequency. The shape of the primary velocity profile is only a weak function of frequency in this region [equation (4)]. In the neighbourhood of $\gamma = 1{\cdot}5$, the growth-rate surface $\mathrm{Re}\{\sigma_1\} = \mathrm{Re}\{\sigma_1(\alpha, \mathscr{R}_p\sqrt{\delta})\}$ is not unimodal. The critical curves are not smooth (figures 1 and 2), since the most dangerous mode of response changes at a frequency just exceeding $\gamma = 1{\cdot}5$. Both responses are, however, synchronous ($\omega_1 = 0$). This phenomenon will be discussed later, in §4.2. As the frequency increases further, the Stokes layer becomes confined to a region close to the inner cylinder, and the critical parameters are expected to become independent of gap width. It follows that

$$\mathscr{R}_p\sqrt{\delta} \sim \gamma^{\frac{3}{2}}, \quad \alpha \sim \gamma.$$

It was found numerically that for $\gamma \geqslant 8{\cdot}0$

$$\mathscr{R}_p\sqrt{\delta} = (15{\cdot}3 \pm 0{\cdot}05)\,\gamma^{\frac{3}{2}}, \quad \alpha = (0{\cdot}85 \pm 0{\cdot}01)\,\gamma.$$

The error reflects the uncertainty in determining that the asymptotic region has been reached and that the critical values have converged at the large values of N required.

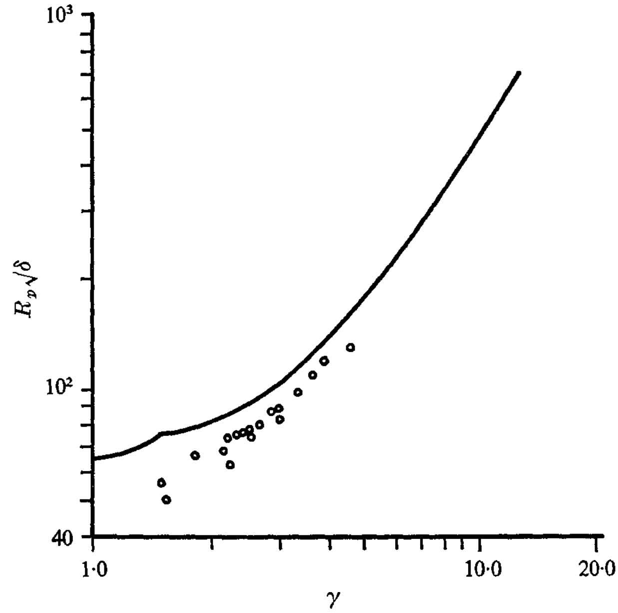

FIGURE 1. Modulation with zero mean. Critical Taylor number $\mathscr{R}_p\sqrt{\gamma}$ as a function of frequency. Data are from Thompson (1968) for $\delta = 0{\cdot}444$, $R_1 = 6{\cdot}0275$ cm.

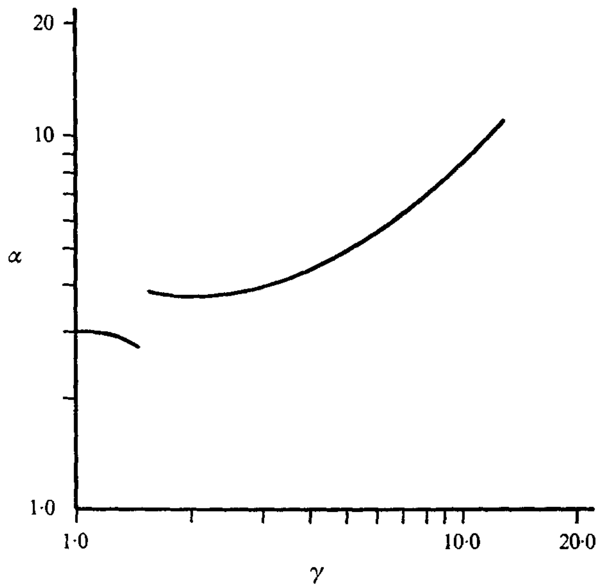

FIGURE 2. Modulation with zero mean. Critical wavenumber α as a function of frequency.

Figure 1 shows a comparison of the theoretical predictions with the data of Thompson (1968), obtained by flow visualization. For $\gamma \geqslant 2{\cdot}0$ the discrepancy between theory and observation is of the order of 10–15 %. The overall trend seems satisfactory. There appears to be some evidence for a change in response, but this is hardly conclusive.

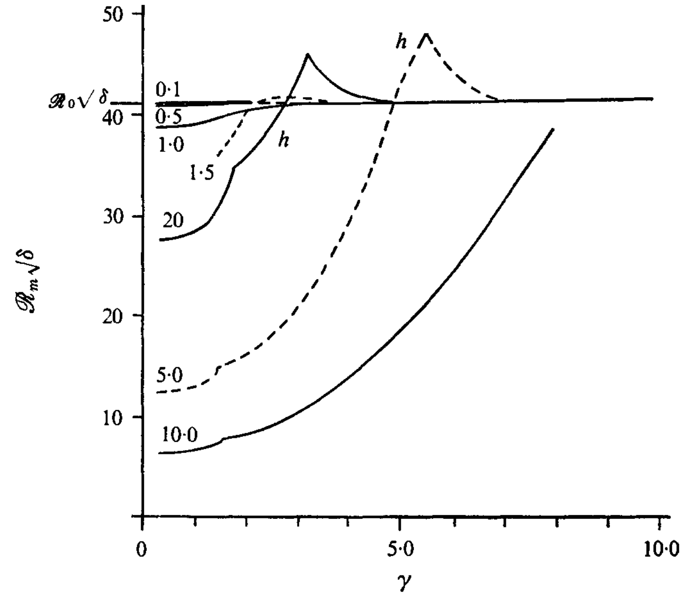

FIGURE 3. Modulation with non-zero mean. Critical Taylor number $\mathscr{R}_m\sqrt{\delta}$ as a function of frequency with the amplitude ratio ϵ as a parameter. h denotes half-frequency response.

For $\gamma < 2{\cdot}0$ the agreement is poor. The experimental values appear to be asymptotic to the critical value $\mathscr{R}_0\sqrt{\delta}$ for steady flow as $\gamma \to 0$. This is, in fact, the strong energy bound in the low frequency limit. It is expected that the linear analysis will be a poor predictive technique (at low frequencies); see Rosenblat & Herbert (1970). The analysis defines a flow to be linearly stable (unstable) if decay (growth) occurs from cycle to cycle. At low frequencies excessive growth may occur during part of a cycle and the class of disturbances for which linear theory is expected to remain valid becomes vanishingly small. Thus even if disturbances decay from cycle to cycle, or if growth observed during one cycle is independent of that observed in adjacent cycles, growth during part of a cycle is likely to be so large that secondary motion will be observed experimentally. Hence instability theory based on a periodicity criterion is likely to be inappropriate at low frequencies (Homsy 1974). If more reliable estimates are needed it is necessary to apply strong energy methods or to solve the complete nonlinear problem.

3.3. *Flows with non-zero mean*

The results as a function of frequency with amplitude ratio as a parameter are presented in figures 3–6. Figure 3 is a plot of the Taylor number $\mathscr{R}_m\sqrt{\delta}$ based on the mean angular velocity of the inner cylinder. Figure 4 presents the predictions in terms of the Taylor number $\mathscr{R}\sqrt{\delta}$ based on the average angular velocity of the inner cylinder, $(\Omega_m^2 + \Omega_p^2)^{\frac{1}{2}}$; this allows comparison with the behaviour for modulation with zero mean and in addition separates the results for small amplitude ratios.

For small amplitude ratios, $\epsilon \leqslant 1{\cdot}0$, the stability characteristics are dominated by the mean rotation. The critical Taylor number increases monotonically with

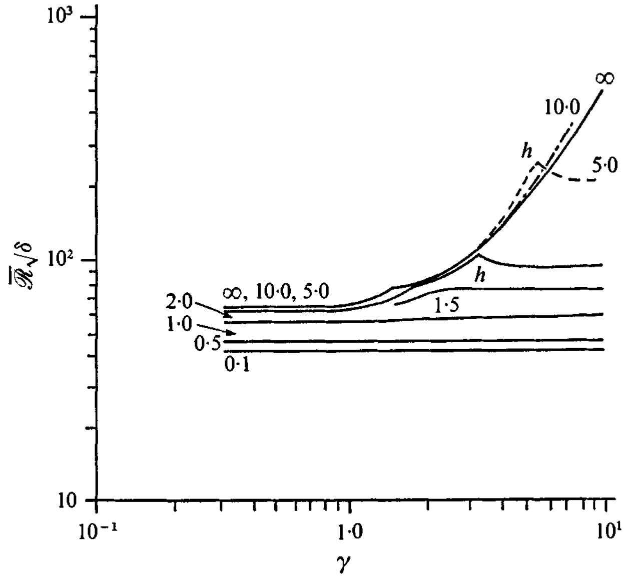

FIGURE 4. Modulated circular Couette flow. Critical Taylor number $\overline{\mathscr{R}}\sqrt{\delta}$ as a function of frequency with the amplitude ratio ϵ as a parameter. h denotes half-frequency response.

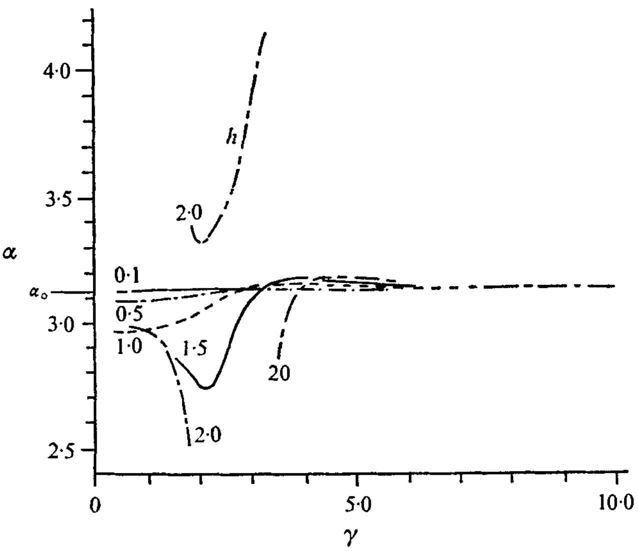

FIGURE 5. Modulated circular Couette flow. Critical wavenumber α as a function of frequency for $\epsilon \leqslant 2\cdot0$. h denotes half-frequency response.

frequency, approaching the steady value asymptotically (see figures 3 and 4). The response is synchronous and modulation destabilizing, the degree of destabilization increasing with modulation amplitude. The reduction of the critical Taylor number $\mathscr{R}_m\sqrt{\delta}$ below the steady value $\mathscr{R}_0\sqrt{\delta}$ is most pronounced in the low frequency limit (see table 2). For $\epsilon \leqslant 0\cdot5$ the effect is negligible from an experimental viewpoint.

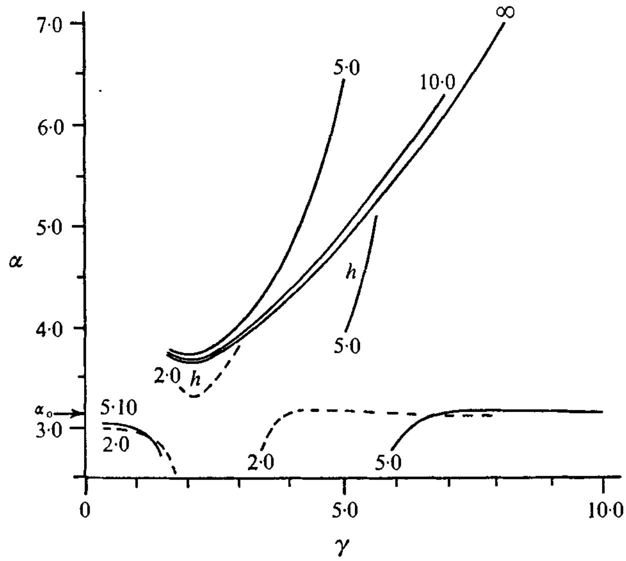

FIGURE 6. Modulated circular Couette flow. Critical wavenumber α as a function of frequency for $\epsilon \geqslant 2{\cdot}0$. h denotes half-frequency response.

ϵ	α	$\mathscr{R}_m\sqrt{\delta}$	$(\mathscr{R}_0\sqrt{\delta})/(\mathscr{R}_m\sqrt{\delta})$	$(\mathscr{R}_0\sqrt{\delta})/(1+\epsilon)$
0·1	3·124	41·186	1·0003	37·45
0·19	3·121	41·152	1·001	34·62
0·25	3·117	41·116	1·002	32·96
0·5	3·088	40·834	1·009	27·47
1·0	2·962	38·743	1·06	20·60
2·0	2·987	27·493	1·50	13·73
5·0	3·030	12·315	3·35	6·87
10·0	3·037	6·261	6·58	3·75
	3·039	62·963†	0·65‡	41·198§

† $\mathscr{R}_p\sqrt{\delta}$. ‡ $(\mathscr{R}_0\sqrt{\delta})/(\mathscr{R}_p\sqrt{\delta})$. § $\mathscr{R}_0\sqrt{\delta}$.

TABLE 2. Critical parameters in the low frequency limit $\gamma = 0{\cdot}3125$, $N = 3$. The values for the steady problem have been taken to be $\mathscr{R}_0\sqrt{\delta} = 41{\cdot}198$, $\alpha_0 = 3{\cdot}126$.

The critical wavenumber is less than the steady value at low frequencies. For $\epsilon \leqslant 1{\cdot}0$ the wavenumber then increases slowly with frequency, passes through a maximum at a frequency γ_a and then decreases to the steady value as frequency increases. For larger amplitude ratios, similar behaviour is observed when the frequency is sufficiently high that instability is primarily associated with the mean rotation. Table 3 gives approximate values of γ_a.

For large modulation amplitudes, $\epsilon \geqslant 5{\cdot}0$, it is apparent from figures 4 and 6 that at low and intermediate frequencies the stability behaviour is dominated by the oscillatory component, and the system behaves effectively as a modulated flow with zero mean.

The critical curves for $\epsilon = 5{\cdot}0$ and $10{\cdot}0$, like that for modulation with zero mean, show a change from one synchronous response to another at $\gamma \approx 1{\cdot}5$.

ϵ	γ_a
0·1	3·75–4·0
0·5	3·75–4·0
1·0	3·75
1·5	4·0
2·0	4·5
5·0	8·0

TABLE 3. Approximate values of frequency γ_a

For $\gamma < 1{\cdot}5$, the critical wavenumber α and Taylor number vary only slightly with amplitude ratio (figures 4 and 6).

Figure 3 shows that, for $\epsilon = 5{\cdot}0$, as the frequency is increased the Taylor number $\mathscr{R}_m\surd\delta$ increases also; eventually the mean rotation alone would be unstable for $4{\cdot}75 < \gamma < 5{\cdot}0$. At $\gamma = 5{\cdot}0$ the response becomes half-frequency ($\omega_1 = \frac{1}{2}\omega$), corresponding to a parametric stabilization of the mean rotation, and the rate of increase of stabilization with frequency is slowed. The critical Taylor number and wavenumber now have values intermediate between that corresponding to instability arising in the Stokes layer and that due to an unstable mean rotation. Between $\gamma = 5{\cdot}5$ and $\gamma = 5{\cdot}75$, the response reverts to synchronous. The wavenumber is now characteristic of an unstable mean rotation, the Stokes layer having too much of a boundary-layer character to affect stability substantially. The critical Taylor number $\mathscr{R}_m\surd\delta$ then decreases to a value less than the steady value $\mathscr{R}_0\surd\delta$, subsequently approaching $\mathscr{R}_0\surd\delta$ asymptotically. The frequencies at which the response changes from synchronous to half-frequency (and back) have not been determined with great accuracy since calculations were generally performed at intervals of frequency of $\Delta\gamma = 0{\cdot}25$. More extensive calculations require extensive computer time since the growth-rate surface

$$\mathrm{Re}\,(\sigma_1) = \mathrm{Re}\,\{\sigma_1(\alpha, \mathscr{R}\surd\delta)\}$$

has three local maxima, one corresponding to a half-frequency response and two to synchronous responses (one associated with instability arising in the Stokes layer and the other with an unstable mean rotation). Thus, in this region it is necessary, in general, to investigate fully at least two maxima.

Similar behaviour is expected for $\gamma = 10{\cdot}0$, though at higher frequencies. The calculations, however, have not been pursued further.

Amplitude ratios of $\epsilon = 1{\cdot}5$ and $2{\cdot}0$ show a steady transition from behaviour dominated by the mean rotation ($\epsilon \leqslant 1{\cdot}0$) to behaviour dominated by the oscillatory shear flow ($\epsilon \geqslant 5{\cdot}0$).

For $\epsilon = 2{\cdot}0$ the response is synchronous until $\gamma = 1{\cdot}85$, when a subharmonic resonance is possible, which slows the rate of increase of Taylor number with frequency (figure 3). The mean rotation alone would become unstable for $2{\cdot}75 < \gamma < 3{\cdot}0$. At $\gamma = 3{\cdot}25$ the response becomes synchronous, and associated with instability of the mean rotation. The wavenumber is plotted in both figure 5 and figure 6 to allow comparison with the values for both smaller and larger amplitude ratios.

ϵ	γ_1	$[\mathscr{R}_m\sqrt{\delta}]_{\gamma_1}$	$[\mathscr{R}_m\sqrt{\delta}]_{\gamma_2}/\mathscr{R}_0\sqrt{\delta}$	γ_2
1·5	3·0	41·554	1·01	5·0
2·0	3·25	46·515	1·13	7·0
5·0	5·5–5·75	46·971	1·14	9·0

TABLE 4. Approximate frequency γ_1 at maximum stabilization, corresponding enhancement relative to the steady problem, and approximate values of frequency γ_2

For $\epsilon = 1{\cdot}5$ the wavenumber initially decreases with increasing frequency, which is similar to the behaviour observed for $\epsilon = 2{\cdot}0$. The oscillatory component of flow is not, however, strong enough to drive a parametric resonance and the response remains synchronous. The wavenumber then increases to a value characteristic of instability of the mean rotation. Weak stabilization is observed for $2{\cdot}25 < \gamma < 4{\cdot}0$.

For $\epsilon \geqslant 1{\cdot}5$ some degree of stabilization is predicted (figure 3). The approximate frequency γ_1 at which maximum stabilization occurs and the corresponding fractional enhancement of stability are given in table 4. The effect may be seen to be fairly small ($\leqslant 15\,\%$). As the frequency increases from γ_1, $\mathscr{R}_m\sqrt{\delta}$ decreases to a minimum which is less than the steady value $\mathscr{R}_0\sqrt{\delta}$, at a frequency γ_2, and then increases monotonically to $\mathscr{R}_0\sqrt{\delta}$ as the frequency is increased further. This behaviour is not particularly apparent from figure 3, so the approximate values of γ_2 are given in table 4.

We observed from tables 3 and 4 that behaviour manifest at a particular amplitude ratio is usually apparent at higher values of the frequency as the amplitude ratio increases.

No stabilization is observed for $\epsilon \leqslant 1{\cdot}0$. Thus it appears that, for modulation to stabilize the mean rotation, the angular velocity of the inner cylinder must be negative during part of the cycle.

Subharmonic responses are found only for $\epsilon \geqslant 2{\cdot}0$ (no calculations were performed for $1{\cdot}5 \leqslant \epsilon \leqslant 2{\cdot}0$). The oscillation must be significant for resonance phenomena to become apparent.

4. Comparison with earlier work

4.1. *Comparison with experiment*

Thompson (1968) remarks that at low frequencies modulated flows with non-zero mean become unstable 'about as soon as'

$$\mathscr{R}_m = \mathscr{R}_0/(1+\epsilon). \tag{20}$$

As remarked earlier, the corresponding result for flows with zero mean is

$$\mathscr{R}_p = \mathscr{R}_0. \tag{21}$$

At low frequencies the velocity profile is given approximately by (4). If $\mathscr{R}_0^E$ is the critical energy bound for the stability of the steady flow to axisymmetric disturbances in the narrow-gap limit, it follows that the modulated flow will be strongly stable (to axisymmetric disturbances) if

$$|\mathscr{R}_m^E(1+\epsilon\cos\omega\tau)| \leqslant \mathscr{R}_0^E,$$

i.e.

$$\mathscr{R}_m^E \leqslant \mathscr{R}_0^E/(1+\epsilon),$$

for flows with non-zero mean or if

$$|\mathscr{R}_p^E\cos\omega\tau| \leqslant \mathscr{R}_0^E,$$

i.e.

$$\mathscr{R}_p^E \leqslant \mathscr{R}_0^E,$$

for flows with zero mean. These predictions have been confirmed by calculations using the complete profile (Riley & Laurence 1976), and are found to be accurate for $\gamma \leqslant 2{\cdot}0$. Thus criteria (20) and (21) are the strong energy bounds if we equate $\mathscr{R}_0^E$ to $\mathscr{R}_0$.

The linear and energy bounds for the steady problem may be shown to be very nearly equal when disturbances are assumed to be axisymmetric and of finite though limited size if use is made of an 'optimal coupling constant' in deriving the energy equalities (Joseph & Hung 1971). Thus we may consider (20) and (21) as being consistent with strong energy theory, since the arguments made above are still valid for the modified energy method.

A comparison of (20) and (21) with the low frequency linear predictions is made in table 2. It can be seen that

$$\mathscr{R}_m\surd\delta \geqslant \mathscr{R}_0\surd\delta/(1+\epsilon), \quad \mathscr{R}_p\surd\delta \geqslant \mathscr{R}_0,$$

as required.

At higher frequencies Thompson states that flow with non-zero mean becomes unstable when

$$\mathscr{R}_m\surd\delta \approx \mathscr{R}_0\surd\delta.$$

Hence these results of Thompson are consistent with the present linear theory for small amplitude ratios. He does not, however, state the values of the amplitude ratio used in his experiments.

The most extensive experimental investigation of the stability of modulated Couette flow is due to Donnelly (1964). It is of interest to discuss the instability criterion he used if we are to make a meaningful comparison of his results and the theory (cf. Homsy 1974).

Donnelly used an apparatus which produced an electrical signal proportional to the radial perturbation of the flow. The signal was integrated over one cycle; we shall refer to this quantity as the amplitude. In the absence of instability, the amplitude should be approximately constant.

Donnelly states that

> The criterion for instability with modulation was taken to be the presence of regular cells revealed by slowly moving the outer cylinder in the axial direction. The cells are always recognised by their periodicity in the axial direction. Under certain circumstances one can find a trace of cell motion as soon as the criterion ...[equation (20)]...is exceeded. However, if Ω_m is increased slightly, the signal does not amplify according to the law discussed by Donnelly (1963).

The law mentioned is that of Landau, i.e. that the amplitude of disturbance secondary motion increases as

$$\text{amplitude} \sim (\mathscr{R}-\mathscr{R}_c)^{\frac{1}{2}}. \tag{22}$$

This relation will hold for a time-periodic primary flow only if bifurcation is one-sided. Joseph (1973) shows that if the response is synchronous (as would be expected for the parameter ranges investigated by Donnelly, $\epsilon \leqslant 0{\cdot}25$, $\gamma \leqslant 1{\cdot}0$) this need not be true. If we note that Donnelly measured amplitude as a function of Reynolds number on a branch which was apparently initially subcritical by linear theory. If bifurcation is indeed one-sided he was presumably measuring disturbance growth on a quasi-steady profile, in which case growth during one cycle might be entirely independent of that in successive cycles. Such a motion would not be truly periodic. If bifurcation is two-sided (22) is not applicable and Donnelly may well have been performing experiments on a branch that was not tangential to the linear solution, or the motion may have been quasi-static as discussed above. For either two-sided or one-sided bifurcation (the latter is expected here, Hall 1975) analysing data in terms of (22) need not determine the correct critical parameters if, as in Donnelly's data, finite amplitude radial motion was observed in situations he considered stable.

There seems, therefore, to be evidence of instability, i.e. 'transient vortices', at Reynolds numbers less than those taken to be the critical values by Donnelly. The vortices first occur at Reynolds numbers given by (20), and this interpretation of his data is consistent with Thompson's remarks and strong energy theory.

4.2. *Comparison with recent theoretical work*

Hall (1975, 1976) analysed the stability of modulated Couette flow with non-zero mean. The results of his analysis reinforce the conclusions of this research, suggesting that modulation has a destabilizing effect.

In the first paper (Hall 1975), two asymptotic cases are reported: the limit as $\gamma \rightarrow 0$ with γ^2/ϵ fixed, and the limit as $\gamma \rightarrow \infty$ with ϵ arbitrary. The low frequency results can be written in the notation of this paper as

$$\mathscr{R}_m^2\delta = \mathscr{R}_0^2\delta - 104{\cdot}3\epsilon^2 + 3{\cdot}4\epsilon^2\gamma^4 + O(\epsilon^4, \epsilon^2\gamma^8).$$

This is wholly consistent with the results given in figure 4. Hall's high frequency results (for large γ with ϵ arbitrary) can be written as

$$\mathscr{R}_m\sqrt{\delta} = (\mathscr{R}_0\sqrt{\delta})\left(1 - 3{\cdot}06\times10^3\left(1-\frac{12{\cdot}25}{\gamma}\right)\frac{\epsilon^2}{\gamma^6}\right).$$

In the second paper (1976) Hall recovers these results in the limit $\epsilon \rightarrow 0$ and sharpens the high frequency result, resolving some difference with our calculations. Modulation clearly destabilizes the flow, the destabilization decreasing with an increase in frequency in accord with the low amplitude results given in figure 3.

4.3. *On the mechanism of instability*

There seems little doubt that a centrifugal driving force is the primary cause of instability. Thompson (1968) found experimentally that steady rotation of the outer cylinder stabilizes modulation of the inner with zero mean, as one would suspect if this were the case.

Rosenblat (1968), on the basis of an inviscid analysis of modulated Couette flow, suggests that a radial phase differential may cause instability, a mechanism independent of factors such as centrifugal instability. Von Kerczek & Davis (1974) note that the inviscid limit is not well defined for flows whose velocity profiles depend explicitly on the kinematic viscosity, in particular for modulated flows. They show, in addition, that a plane Stokes layer is linearly stable at least for $\mathscr{R}_p \leqslant 800\gamma$. Hence a phase differential alone is probably insufficient to cause instability.

The increasing radial phase differential may possibly influence the change in the most dangerous response as frequency is increased from low values ($\gamma < 1{\cdot}0$), the change from one synchronous response to another at $\gamma = 1{\cdot}5$ for modulation with zero mean being a possible example. This appears, however, to be a difficult notion to formulate quantitatively.

The disturbance kinetic energy was examined in this frequency range to investigate the response. For modulation with zero mean, pulses occur on both forward and backward swings. Similar behaviour was observed for frequencies both less than and greater than $\gamma = 1{\cdot}5$. For larger amplitude ratios $\epsilon \geqslant 5{\cdot}0$, pulses occur both on backward and forward swings, though the mean rotation reduces the pulse intensity on the backward swing. Thus the kinetic energy shows no marked variation as the response changes at $\gamma = 1{\cdot}5$.

The details of the velocity field do change, however. For flow with zero mean and for $\gamma < 1{\cdot}5$, the radial and axial components of the disturbance pulse in different directions on successive swings and appear to have zero mean. The azimuthal component, while pulsing twice during a cycle, does so in the same direction. For $\gamma > 1{\cdot}5$, the radial and axial components now have the same direction on successive pulses, while the azimuthal component reverses direction on successive swings and appears to have zero mean. This high frequency behaviour has already been noted by Thompson (1968). For large modulation amplitudes $\epsilon \geqslant 5{\cdot}0$ the velocity components pulse in the same sense as for modulation with zero mean at the same frequency. No component now has zero mean, however, as the mean rotation opposes growth on the backward swing. For lower modulation amplitudes $\epsilon \leqslant 2{\cdot}0$ the disturbance kinetic energy pulses only on the forward swing.

5. Conclusions

If we present results in terms of a Taylor number $\mathscr{R}_m\sqrt{\delta}$ based on the mean rotation we may conclude that, in general, modulation is destabilizing. At low modulation amplitudes, $\epsilon \leqslant 1{\cdot}0$, the behaviour is dominated by instability of the mean rotation. For large modulation amplitudes, $\epsilon \geqslant 5{\cdot}0$, instability associated with the oscillatory shear is dominant until the frequency is sufficiently high that the Stokes layer is too thin substantially to affect disturbance growth. If the velocity of the inner cylinder is negative during part of the cycle, weak stabilization is possible. If, however, $\epsilon \geqslant 2{\cdot}0$, a degree of stabilization is possible, and is associated with a parametric resonance of the oscillatory shear and the mean rotation.

Results in terms of the Taylor number $\bar{\mathscr{R}}\sqrt{\delta}$ or $(1+\epsilon)\mathscr{R}_m\sqrt{\delta}$ show that, for a modulated system, larger angular accelerations may be attained than in the corresponding steady problem before instability is manifest. Thus, in this sense, modulation may be considered stabilizing.

Both linear and energy theories of the stability of time-periodic motions seem well developed. There appears, however, to be a paucity of data for comparison with theory. The data available suggest that strong energy estimates (though not necessarily in a global sense; cf. Joseph & Hung 1971) are more reliable at low frequencies than either linear or mean energy bounds, where by mean energy theory we mean an energy method based on a periodicity criterion (Davis & von Kerczek 1973; Homsy 1974). At higher frequencies, say $\gamma \geqslant 2{\cdot}0$ for the present problem, linear theory appears to be in reasonable agreement with experiment.

The authors would like to express their appreciation to the University of Massachusetts Computing Centre for grants without which the computations reported in this paper would not have been possible, and to Dr Philip Hall for copies of his manuscript prior to publication.

Appendix

The Jacobi polynomials chosen satisfy the following normalization conditions (Morse & Feshbach 1953, p. 780):

$$\int_0^1 x^2(1-x)^2 \mathscr{J}_n(x)\mathscr{J}_m(x)\,dx = \frac{\delta_{nm}\,n!\,4}{(5+2n)(4+n)!}$$

$$\int_0^1 x(1-x)\,\tilde{\mathscr{J}}_n(x)\,\tilde{\mathscr{J}}_m(x)\,dx = \frac{\delta_{nm}\,n!}{(3+2n)(2+n)!}.$$

The sets $\{\phi_j\}_{j=1}^{\infty}$ and $\{\tilde{\phi}_j\}_{j=1}^{\infty}$ defined by (14) may be shown to be complete in $L^2[0,1]$ (Riley 1975). The disturbance equations ($12a, b$) may be written as

$$-\mathscr{L}\partial u/\partial\tau = -\mathscr{L}^2 u + \alpha\mathscr{R}(2\delta)^{\frac{1}{2}}\,Vv, \tag{A 1}$$

$$\partial v/\partial\tau = \mathscr{L}v - \alpha\mathscr{R}(2\delta)^{\frac{1}{2}}(DV)\,u. \tag{A 2}$$

Define

$$A_{ij} = -\int_0^1 \phi_i\mathscr{L}\phi_j\,dx, \quad \tilde{A}_{ij} = \int_0^1 \tilde{\phi}_i\tilde{\phi}_j\,dx,$$

$$B_{ij} = -\int_0^1 \phi_i\mathscr{L}^2\phi_j\,dx, \quad \tilde{B}_{ij} = \int_0^1 \tilde{\phi}_i\mathscr{L}\tilde{\phi}_j\,dx,$$

$$C_{ij} = \int_0^1 \phi_i\,V\tilde{\phi}_j\,dx, \quad \tilde{C}_{ij} = -\int_0^1 \tilde{\phi}_i(DV)\,\phi_j\,dx.$$

All integrations may be performed analytically. It is, however, more convenient to evaluate the contributions to $\mathbf{C}$ and $\tilde{\mathbf{C}}$ arising from the oscillatory shear flow using Gaussian quadrature. After the necessary integrations have been performed (A1) and (A2) are transformed into

$$\begin{bmatrix} \mathbf{A} & 0 \\ 0 & \tilde{\mathbf{A}} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{a}} \\ \dot{\mathbf{b}} \end{bmatrix} = \begin{bmatrix} \mathbf{B} & \alpha\mathscr{R}(2\delta)^{\frac{1}{2}}\mathbf{C} \\ \alpha\mathscr{R}(2\delta)^{\frac{1}{2}}\tilde{\mathbf{A}} & \tilde{\mathbf{B}} \end{bmatrix} \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}, \tag{A 3}$$

where a dot denotes differentiation with respect to τ.

The matrix containing $\mathbf{A}$ in (A 3) is positive definite and therefore invertible. Equation (A 3) may then be written as

$$\begin{bmatrix} \dot{\mathbf{a}} \\ \dot{\mathbf{b}} \end{bmatrix} = \mathbf{G} \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix},$$

where

$$\mathbf{G} = \begin{bmatrix} \mathbf{A}^{-1} & 0 \\ 0 & \tilde{\mathbf{A}}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{B} & \alpha\mathscr{R}(2\delta)^{\frac{1}{2}}\mathbf{C} \\ \alpha\mathscr{R}(2\delta)^{\frac{1}{2}}\tilde{\mathbf{C}} & \tilde{\mathbf{B}} \end{bmatrix}.$$

The expansion functions $\{\phi_i\}_{j=1}^{\infty}$ and $\{\tilde{\phi}_j\}_{j=1}^{\infty}$ have some degree of orthogonality (they are in fact minimal), and hence are expected to give rise to a reasonably stable numerical scheme (see Mikhlin 1971, p. 132). While testing sections of the program, the steady problem was solved by the standard eigenvalue method using the principle of the exchange of stabilities. Values of N up to 20 were taken without evidence of numerical instability.

In reservation we note that convergence appears to be slightly slower than that obtained with the functions used by Yih & Li (1972) or Rosenblat & Tanaka (1971), at least for the steady problem.

REFERENCES

Chandrasekhar, S. 1961 *Hydrodynamic and Hydromagnetic Stability*. Oxford: Clarendon Press.

Coddington, E. A. & Levinson, N. 1955 *Theory of Ordinary Differential Equations*. McGraw-Hill.

Conrad, P. W. & Criminale, W. O. 1965 *Z. angew. Math. Phys.* **16**, 569.

Davis, S. H. 1970 *J. Fluid Mech.* **45**, 33.

Davis, S. H. 1976 *Ann. Rev. Fluid Mech.* **8**, 57.

Davis, S. H. & von Kerczek, C. 1973 *Arch. Rat. Anal.* **52**, 112.

Donnelly, R. J. 1963 *Phys. Rev. Lett.* **10**, 282.

Donnelly, R. J. 1964 *Proc. Roy. Soc.* A **281**, 130.

Francis, J. G. F. 1961 *Computer J.* **4**, 265–271, 332–345.

Hall, P. 1975 *J. Fluid Mech.* **67**, 29.

Hall, P. 1976 *J. Fluid Mech.* (to appear).

Homsy, G. M. 1974 *J. Fluid Mech.* **62**, 387.

Iooss, G. 1972 *Comptes Rendus*, A **275**, 935.

Jacoby, S. L. S., Kowalik, J. S. & Pizzo, J. T. 1972 *Iterative Methods for Non-linear Optimization Problems*. Prentice-Hall.

Joseph, D. D. 1972 Remarks about bifurcation and stability of quasi-periodic solutions which bifurcate from periodic solutions of the Navier–Stokes equations. In *Non-Linear Problems in the Physical Sciences and Biology. Lecture Notes in Mathematics*, no. 322, p. 130. Springer.

JOSEPH, D. D. & HUNG, W. 1971 *Arch. Rat. Mech. Anal.* **44**, 1.
KERCZEK, C. VON & DAVIS, S. H. 1974 *J. Fluid Mech.* **62**, 753.
KRUGER, E. R., GROSS, A. & DIPRIMA, R. C. 1966 *J. Fluid Mech.* **24**, 521.
LAPIDUS, L. & SEINFELD, J. H. 1971 *Numerical Solutions of Ordinary Differential Equations*. Academic.
MEISTER, B. 1963 *Arch. Rat. Mech. Anal.* **14**, 81.
MEISTER, B. & MÜNZER, W. 1966 *Z. angew. Math. Phys.* **17**, 537.
MIKHLIN, S. G. 1971 *The Numerical Performance of Variational Methods*. Wolters Noordhoff.
MORSE, P. M. & FESHBACH, H. 1953 *Methods of Theoretical Physics*, vol. I. McGraw-Hill.
RALSTON, A. 1965 *A First Course in Numerical Analysis*. McGraw-Hill.
RILEY, P. J. 1975 Ph.D. thesis, Dept. Chemical Engng, University of Massachusetts.
RILEY, P. J. & LAURENCE, R. L. 1976 *J. Fluid Mech.* (to appear).
ROSENBLAT, S. 1968 *J. Fluid Mech.* **33**, 321.
ROSENBLAT, S. & HERBERT, D. M. 1970 *J. Fluid Mech.* **43**, 385.
ROSENBLAT, S. & TANAKA, G. A. 1971 *Phys. Fluids*, **14**, 1319.
RUELLE, D. & TAKENS, F. 1971 *Comm. Math. Phys.* **20**, 167.
SERRIN, J. 1959 *Arch. Rat. Mech. Anal.* **1**, 1.
SERRIN, J. 1960 *Arch. Rat. Mech. Anal.* **3**, 120.
TAYLOR, G. I. 1923 *Phil. Trans.* A **223**, 289.
THOMPSON, R. 1968 Ph.D. thesis, Dept. of Meteorology, M.I.T.
WILKINSON, J. H. 1965 *Computer J.* **8**, 77.
YIH, C. S. & LI, C. H. 1972 *J. Fluid Mech.* **54**, 143.
YUDOVITCH, V. 1970 *Sov. Math. Dokl.* **11**, 1473.